# THE TEMPERATURE FIELD IN A HOMOGENEOUS MEDIUM HEATED BY AN ELECTRIC CURRENT UNDER CONDITIONS OF SPHERICAL SYMMETRY

M. A. Kaganov and V. A. Yangarber

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 110-113, 1965

Consider the temperature field in a homogeneous medium near a spherical electrode heated by a constant current. The temperature depends on the Joule heat liberated in the medium, in the electrode, and at the electrode-medium interface (due to contact resistance). A problem of this kind is of interest in the analysis of contact welding, electric grounding devices, and certain other applications. Until now this problem has been solved only approximately, for specific numerical examples [1].

The change of temperature at the electrode-medium interface can be used to determine the thermophysical properties of the medium [2, 3]. The equations for the thermal conductivity and diffusivity do not, however, take into account the heat liberated inside the investigated sample. This can lead to considerable error, particularly in the case of materials with relatively high resistivity (semiconductors).

The temperature at a distance r from the center of the electrode and time t from the instant of switching on the current I is determined by the heat conduction equation

$$\frac{1}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{I^2\rho}{16\pi^2\lambda r^4} \tag{1}$$

with the initial condition

$$T(r, t)\big|_{t=0} = 0 \tag{2}$$

and boundary conditions

$$q = G\frac{\partial T}{\partial t}\Big|_{r=r_0} - 4\pi\lambda r_0^2\frac{\partial T}{\partial r}\Big|_{r=r_0}, \qquad T(r, t) \to 0 \qquad (r \to \infty). \tag{3}$$

Here $r_0$ is the radius of the electrode, $\lambda$ and k are the thermal conductivity and diffusivity, respectively, $\rho$ is the electric resistivity of the medium, G is the total heat capacity of the electrode, and q is the power dissipated inside the electrode and at its surface.

The thermal conductivity of the material is assumed to be high enough for the temperature gradient along the radius of the electrode to be neglected.

We shall solve Eq. (1) by the method of sources. Substituting

$$\frac{kt}{r_0^2} = \tau, \qquad \frac{r}{r_0} - 1 = x, \qquad u = r_0(x+1)T + \frac{A}{2(x+1)},$$

into (1)-(3), we obtain

$$\begin{gathered}\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x, \tau)\big|_{\tau=0} = \frac{A}{2(x+1)} \equiv f(x), \\ \left(\frac{\partial u}{\partial x} - u\right)\Big|_{x=0} = -q^* + \gamma\frac{\partial u}{\partial \tau}\Big|_{x=0}, \qquad u(x, \tau) \to 0 \quad (x \to \infty) \\ A = \frac{I^2\rho}{16\pi^2\lambda r_0} = \frac{Q}{4\pi\lambda}, \qquad q^* = \frac{q+Q}{4\pi\lambda}, \qquad \gamma = \frac{Gk}{4\pi\lambda r_0^3} = \frac{c_1}{3c_2}, \qquad Q = \frac{I^2\rho}{4\pi r_0}.\end{gathered} \tag{4}$$

Here Q is power dissipated in the medium, and $c_1$, $c_2$ are the specific heats per unit volume of the electrode and of the medium, respectively.

We seek a solution to (4) in the form (cf. [4])

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}}\int_0^\infty \left[f(\xi)\exp\left\{-\frac{(x-\xi)^2}{4\tau}\right\} + f(-\xi)\exp\left\{-\frac{(x+\xi)^2}{4\tau}\right\}\right]d\xi. \tag{5}$$

Here $f(-\xi) = \varphi(\xi)$ is a continuous continuation of the function $f(\xi)$ on the negative semiaxis. Thus $\varphi(0) = f(0) = 1/2A$. We impose on the function $\varphi$ three additional conditions,

$$\varphi'(0) = -f'(0), \tag{6}$$

$$\varphi'(\xi)\, e^{-\delta\xi^2} = o(1), \qquad \xi\varphi(\xi)\, e^{-\delta\xi^2} = o(1), \qquad (\delta > 0) \text{ when } \xi \to \infty, \tag{7}$$

and, finally, we require that $\varphi(\xi)$ be such that boundary condition (4) for $u(x, \tau)$ is satisfied at $x = 0$. We introduce the functional

$$P(\psi) = \int_0^\infty \exp\left\{-\frac{\xi^2}{4\tau}\right\}\psi(\xi)\, d\xi, \qquad P(1) = \sqrt{\pi\tau}$$

Then, in view of the additivity and uniformity of the functional P under the condition (7) imposed on the function $\psi$, we have

$$\begin{aligned} P(\xi\psi) &= 2\tau\psi(0) + 2\tau P\left(\frac{d\psi}{d\xi}\right), \\ P(\xi^2\psi) &= 2\tau P(\psi) + 4\tau^2 \frac{d\psi(0)}{d\xi} + 4\tau^2 P\left(\frac{d^2\psi}{d\xi^2}\right). \end{aligned} \tag{8}$$

As $f(\xi)$ obviously satisfies condition (7), Eqs. (8) hold if $\psi$ is replaced by $f$ or $\varphi$.

Equations (7), (8) with boundary condition (4) at $x = 0$ and condition (6) yield

$$P(f') - P(\varphi') - P(f + \varphi) = -P(2q^*) + \gamma P(f'' + \varphi''),$$

which holds if $\varphi$ is a solution of the ordinary differential equation

$$L(\varphi) \equiv \varphi'' + \gamma^{-1}\varphi' + \gamma^{-1}\varphi = -f'' + \gamma^{-1}f' - \gamma^{-1}f + 2\gamma^{-1}q^*.$$

We now seek a solution $\varphi(\xi)$ in the form

$$\varphi(\xi) = -f(\xi) + 2q^* + h(\xi). \tag{9}$$

In this case the function $h(\xi)$ is a solution of the problem

$$L(h) = \frac{2}{\gamma} f'(\xi), \qquad h(0) = A - 2q^* = -q^{**} = -\frac{2q + Q}{4\pi\lambda}, \qquad h'(0) = 0. \tag{10}$$

On can easily see that

$$h(\xi) = \frac{q^{**}}{\gamma\Delta\alpha}\Delta\left(\frac{1}{\alpha}e^{\alpha\xi}\right) + \frac{2}{\gamma\Delta\alpha}\int_0^\xi f'(z)\,\Delta e^{\alpha(\xi - z)}\, dz, \tag{11}$$

where $\alpha_1$ and $\alpha_2$ are the roots of the equation $\gamma\alpha^2 + \alpha + 1 = 0$ ($\gamma > 0$), and the symbol $\Delta$ is defined by the expression $\Delta y(\alpha) = y(\alpha_1) - y(\alpha_2)$ for any function $y(\alpha)$.

Substituting (11) into (9) and (5), we obtain

$$\begin{aligned} u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}}\Bigg\{&\int_0^\infty f(\xi)\exp\left[-\frac{(x-\xi)^2}{4\tau}\right]d\xi + \int_0^\infty\Bigg[\frac{2}{\gamma\Delta\alpha}\Delta\left(\int_0^\xi f'(z)\, e^{\alpha(\xi - z)}\, dz\right) + \\ &+ \frac{q^{**}}{\gamma\Delta\alpha}\Delta\left(\frac{1}{\alpha}e^{\alpha\xi}\right) - f(\xi) + 2q^*\Bigg]\exp\left[-\frac{(x+\xi)^2}{4\tau}\right]d\xi\Bigg\}. \end{aligned}$$

From this we obtain the final expression for the temperature field in a homogeneous medium heated by a constant current

$$\begin{aligned} T(x, \tau) = \frac{1}{4\pi\lambda r_0(x+1)}\Bigg\{&-\frac{Q}{2(x+1)} + (q + Q)\left[1 - \Phi\left(\frac{x}{2\sqrt{\tau}}\right)\right] + \\ &+ \frac{2q + Q}{2\gamma\Delta\alpha}\Delta\left(\frac{1}{\alpha}e^{\alpha^2\tau - \alpha x}\left[1 - \Phi\left(\frac{x}{2\sqrt{\tau}} - \alpha\sqrt{\tau}\right)\right]\right) + \end{aligned} \tag{12}$$

$$+\frac{Q}{2\sqrt{\pi\tau}}\exp\left\{-\frac{x^2}{4\tau}\right\}\int_0^\infty \exp\left\{-\frac{\xi^2}{4\tau}\right\}\operatorname{sh}\frac{\xi x}{2\tau}\frac{d\xi}{\xi+1}-$$

$$-\frac{Q}{2\gamma\Delta\alpha}\Delta\left(e^{\alpha^2\tau-\alpha x}\int_0^\infty e^{-\alpha\xi}\left[1-\Phi\left(\frac{\xi+x}{2\sqrt{\tau}}-\alpha\sqrt{\tau}\right)\right]\frac{d\xi}{(\xi+1)^2}\right)\Bigg\}. \qquad (12)$$

(cont'd)

Here $\Phi(z)$ is the error function. The integrals on the right-hand side of (12) can easily be computed for specific values of the parameters, and the numerical integration converges rapidly due to the fast decrease of the integrand with increasing $\xi$.

In applications one is often interested in the surface temperature of the electrode, i.e., in the value of the right-hand side of (12) at $r = r_0$. The expression for the temperature $T(r_0, t)$ at the surface of the electrode can be written in simpler form

$$T(r_0,t)=\frac{1}{8\pi\lambda r_0\gamma\Delta\alpha}\left[(q+2Q)\Delta\left(\frac{\mu(\alpha\sqrt{\tau})-1}{\alpha}\right)-\right.$$

$$\left.-Q\Delta\left(e^{\alpha^2\tau}\int_0^\infty e^{-\alpha\xi}\left[1-\Phi\left(\frac{\xi}{2\sqrt{\tau}}-\alpha\sqrt{\tau}\right)\right]\frac{d\xi}{(\xi+1)^2}\right)\right] \qquad (13)$$

$$\mu(z)=\exp\{z^2\}[1+\Phi(z)].$$

Note that when $Q = 0$ ($A = 0$), Eq. (13) is identical with the expression for the temperature of a spherical probe, used for measuring the thermophysical properties of dispersed materials [5].

In the limiting case $t \to \infty$, Eq. (12) yields the steady-state temperature distribution

$$T_*=\frac{1}{4\pi\lambda r}\left[q+Q\left(1-\frac{r_0}{2r}\right)\right].$$

In many cases it is sufficient to consider the nonsteady-state temperature for very short times and the final steady-state temperature only. In these cases we use (13) to make an approximate calculation of $T(r_0, \tau)$ for $\tau \ll 1$ and for $\tau \gg\gg 1$. Thus,

$$T(r_0,\tau)=\frac{\tau}{4\pi\lambda r_0\gamma}\left(q+\frac{4}{3}Q\sqrt{\frac{\tau}{\pi}}\right)+o(\tau^{3/2}) \quad \text{for } \tau\ll 1.$$

When the heat capacity of the electrode is small (e.g., a hollow sphere), for $\gamma \ll \tau \ll 1$ we have

$$T(r_0,\tau)=\frac{\sqrt{\tau}}{2\pi^{3/2}\lambda r_0}\left(q+\frac{1}{2}\sqrt{\pi\tau}\,Q\right)+o(\tau)$$

$$T=\frac{2q+Q}{8\pi\lambda r_0}\left[1-\frac{1}{\sqrt{\pi\tau}}\left(1+\frac{Q}{Q+2q}\right)\right]+o\left(\frac{1}{\tau}\right) \text{ in the case } \tau\gg 1$$

The results obtained above can also be used to calculate the temperature field $T^*(r, t)$ when the current has the form of a square pulse of duration $t_0$. In that case, in view of the additivity of the solutions of the heat conduction equation for constant sources,

$$T^*(r,t)=T(r,t)-T(r,t-t_0) \quad \text{for } t\geq t_0.$$

If, at the same time, $kt_0/r_0^2 \ll 1$, then

$$T^*\approx t_0\frac{\partial T(r,t)}{\partial t}.$$

It is clear, further, that if the current is switched off after the steady state is reached, then the temperature field $T^{**}(r, t)$ will decay according to the law

$$T^{**}r,t)=T_*-T(r,t).$$

Finally, note that our calculation was made for the case when $\Delta\alpha \neq 0$, which does not restrict the generality. For when $\Delta\alpha = 0$ ($\gamma = 1/4$), the solution to problem (10) is the function $c_1e^{\alpha_1\xi}(1 + c_2\xi)$ and the form of (12) will change correspondingly. But we can use the original form of (12) even in the case $\Delta\alpha = 0$, merely by taking the limit $\alpha_1 \to \alpha_2 \to \to -2$. Moreover, solution (12) also includes the case $\gamma = 0$. In that case one should take the limit of (12) for $\gamma \to 0$,

$\alpha_1 \to -1$, $\alpha_2 \to -\infty$. For simplicity, we shall write down the expression for $u(x, \tau)$ for the case $\gamma = 0$:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_0^\infty \left\{ f(\xi) \exp\left[-\frac{(x-\xi)^2}{4\tau}\right] + \left[2\int_0^\xi f'(\xi)\, e^{z-\xi}\, dz + q^{**}e^{-\xi} - \right.\right.$$
$$\left.\left. - f(\xi) + 2q^*\right] \exp\left[-\frac{(x+\xi)^2}{4\tau}\right]\right\} d\xi\,.$$

In particular, at the surface of the electrode ($x = 0$) we have, when $\gamma = 0$,

$$T(r_0,\tau) = \frac{q+Q}{8\pi\lambda r_0}\left[1-\mu(-\sqrt{\tau})\right] - \frac{Q}{8\pi\lambda r_0} e^\tau \int_0^\infty e^\xi \left[1 - \Phi\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\tau}\right)\right] \frac{d\xi}{(\xi+1)^2}\,.$$

REFERENCES

1. N. N. Rykalin, "Theory of the heating of bars by an electric current during resistance butt-welding," in: Thermal Processes in Contact Welding [in Russian], Izd-vo AN SSSR, 1959.

2. M. Cutler, "Thermoelectric measurements at small area contacts," J. Appl. Phys., vol. 32, no. 6, 1961.

3. M. Cutler and G. T. Cheney, "Measurement of thermal conductivity of electrical conductors at high temperatures," J. Appl. Phys., vol. 34, no. 6, 1963.

4. V. I. Smirnov, Course in Higher Mathematics [in Russian], vol. 2, 1956.

5. M. A. Kaganov, "Device for the determination of the thermal properties of soil," Collection of Works on Agricultural Physics [in Russian], no. 5, 1952.

6 April 1964 Leningrad